DERIVE BLACK-SCHOLES PDE

LARRY, ZHIRONG LI

Derive the Black-Scholes equation (just the sketch), and give a few numerical methods to solve this kind of SDE.

Readers are referred to Prof. Peter Forsyth's lecture notes for the numerical methods in solving the SDE.

The basic assumptions for B-S equation is

- The price movement of the underlying follows a GBM process, i.e.,
$$\frac{dS_t}{S_t} = \mu(S_t, t)\,dt + \sigma(S_t, t)\,dW_t$$
and the bond price follows the
$$\frac{dB_t}{B_t} = r dt$$
- There are no transaction costs and any other market frictions.
- All securities are perfectly divisible.
- There are no risk free arbitrage opportunities.
- Short selling is allowed.

Method 1 (Replicating the payoff): To derive the Black-Scholes PDE, one can make the replicating portfolio's worth the same as the current value of the option. Suppose that the replicating portfolio V_t is composed of x_t shares of the underlying and y_t units of the risk-free bond. Thus

$$V_t = x_t S_t + y_t B_t$$

If the portfolio is self-financing, then

$$\begin{aligned} dV_t &= x_t dS_t + y_t dB_t \\ &= x_t S_t \sigma(S_t, t)\,dW_t + (x_t S_t \mu(S_t, t) + y_t r B_t)\,dt \end{aligned}$$

Based on the Itô's lemma,

$$\begin{aligned} dC_t(S_t, t) &= \frac{\partial C_t}{\partial S_t} dS_t + \frac{\partial C_t}{\partial t} dt + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}(dS_t)^2 \\ &= \left(\frac{\partial C_t}{\partial t} + \frac{\partial C_t}{\partial S_t} S_t \mu(S_t, t) + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2} S_t^2 \sigma^2(S_t, t)\right) dt + \frac{\partial C_t}{\partial S_t} S_t \sigma(S_t, t)\,dW_t \end{aligned}$$

Therefore, by equating the above two equations, we can get

$$\frac{\partial C_t}{\partial S_t} = x_t$$

which is the option delta and

$$\frac{C_t - \frac{\partial C_t}{\partial S_t} S_t}{B_t} = y_t$$

Date: June 09, 2012.

Hence

$$\frac{\partial C_t}{\partial t}+\frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}S_t^2\sigma^2\left(S_t,t\right)=y_t r B_t=r\left(C_t-\frac{\partial C_t}{\partial S_t}S_t\right)$$

Rearrange the above terms and simplify, we can get the Black-Scholes equation as

$$\frac{\partial C_t}{\partial t}+r\frac{\partial C_t}{\partial S_t}S_t+\frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}S_t^2\sigma^2\left(S_t,t\right)-rC_t=0$$

The second approach can be achieved by considering the market risk under CAPM.

Method 2 (CAPM): In an equilibrium market, the ROI in excess of the risk-free return for a single stock should be the same as the ROI for the options. By taking the extraneous risk, the investors would expect to be compensated for this accordingly.

The ROI for the stock during dt is $\frac{dS_t}{S_t}=\mu\left(S_t,t\right)dt+\sigma\left(S_t,t\right)dW_t$, hence the expected ROI in excess of the risk free rate r per unit time is $\mu\left(S_t,t\right)-r$ and the instantaneous risk measured by the standard deviation is given by $\sqrt{\frac{\left(\sigma(S_t,t)dW_t\right)^2}{dt}}=\sigma\left(S_t,t\right)$. The sharp ratio for the stock is defined by

$$\frac{\mu\left(S_t,t\right)-r}{\sigma\left(S_t,t\right)}$$

Similarly, the ROI for the option during dt time is $\frac{dC_t}{C_t}=\frac{1}{C_t}\left(\frac{\partial C_t}{\partial S_t}dS_t+\frac{\partial C_t}{\partial t}dt+\frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}\left(dS_t\right)^2\right)$. Hence the expected ROI in excess of the risk free rate r is given by

$$\begin{aligned}&\tfrac{1}{C_t}\left(\tfrac{\partial C_t}{\partial S_t}dS_t+\tfrac{\partial C_t}{\partial t}dt+\tfrac{1}{2}\tfrac{\partial^2 C_t}{\partial S_t^2}\left(dS_t\right)^2-rC_t dt\right)\\ =\;&\tfrac{1}{C_t}\left(\tfrac{\partial C_t}{\partial t}+\tfrac{\partial C_t}{\partial S_t}S_t\mu\left(S_t,t\right)+\tfrac{1}{2}\tfrac{\partial^2 C_t}{\partial S_t^2}S_t^2\sigma^2\left(S_t,t\right)-rC_t\right)dt+\tfrac{1}{C_t}\left(\tfrac{\partial C_t}{\partial S_t}S_t\sigma\left(S_t,t\right)dW_t\right)\end{aligned}$$

As a result, the expected ROI in excess of the risk free return rate r per unit time is derived as

$$\frac{1}{C_t}\left(\frac{\partial C_t}{\partial t}+\frac{\partial C_t}{\partial S_t}S_t\mu\left(S_t,t\right)+\frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}S_t^2\sigma^2\left(S_t,t\right)-rC_t\right)$$

and the risk per unit time is given by

$$\frac{1}{C_t}\left(\frac{\partial C_t}{\partial S_t}S_t\sigma\left(S_t,t\right)\right)$$

Equating the above two sharp ratios for stock and option, we can deduce that

$$\frac{\frac{\partial C_t}{\partial t}+\frac{\partial C_t}{\partial S_t}S_t\mu\left(S_t,t\right)+\frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}S_t^2\sigma^2\left(S_t,t\right)-rC_t}{\frac{\partial C_t}{\partial S_t}S_t\sigma\left(S_t,t\right)}=\frac{\mu\left(S_t,t\right)-r}{\sigma\left(S_t,t\right)}$$

Rearrange the above terms, we can get the following as expected.

$$\frac{\partial C_t}{\partial t}+r\frac{\partial C_t}{\partial S_t}S_t+\frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2}S_t^2\sigma^2\left(S_t,t\right)-rC_t=0$$